

# On the Signed (Total) $k$ -Domination Number of a Graph\*

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## Abstract

Let  $k$  be a positive integer and  $G = (V, E)$  be a graph of minimum degree at least  $k - 1$ . A function  $f : V \rightarrow \{-1, 1\}$  is called a *signed  $k$ -dominating function* of  $G$  if  $\sum_{u \in N_G[v]} f(u) \geq k$  for all  $v \in V$ . The *signed  $k$ -domination number* of  $G$  is the minimum value of  $\sum_{v \in V} f(v)$  taken over all signed  $k$ -dominating functions of  $G$ . The *signed total  $k$ -dominating function* and *signed total  $k$ -domination number* of  $G$  can be similarly defined by changing the closed neighborhood  $N_G[v]$  to the open neighborhood  $N_G(v)$  in the definition. The *upper signed  $k$ -domination number* is the maximum value of  $\sum_{v \in V} f(v)$  taken over all *minimal* signed  $k$ -dominating functions of  $G$ . In this paper, we study these graph parameters from both algorithmic complexity and graph-theoretic perspectives. We prove that for every fixed  $k \geq 1$ , the problems of computing these three parameters are all  $\mathcal{NP}$ -hard. We also present sharp lower bounds on the signed  $k$ -domination number and signed total  $k$ -domination number for general graphs in terms of their minimum and maximum degrees, generalizing several known results about signed domination.

## 1 Introduction

All graphs considered in this paper are simple and undirected. We generally follow [4] for standard notation and terminology in graph theory. Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . The *order* of  $G$  is  $|V(G)|$ . For each vertex  $v \in V(G)$ , let  $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$  and  $N_G[v] = N_G(v) \cup \{v\}$ , which are called the *open neighborhood* and *closed neighborhood* of  $v$  (in  $G$ ), respectively. The *degree* of  $v$  (in  $G$ ) is  $d_G(v) = |N_G(v)|$ . The *minimum degree* of  $G$  is  $\delta(G) = \min_{v \in V(G)} \{d_G(v)\}$ , and the *maximum degree* of  $G$  is  $\Delta(G) = \max_{v \in V(G)} \{d_G(v)\}$ . For an integer  $r$ ,  $G$  is called  *$r$ -regular* if  $\Delta(G) = \delta(G) = r$ , and is called *nearly  $r$ -regular* if  $\Delta(G) = r$  and  $\delta(G) = r - 1$ . For  $S \subseteq V(G)$ ,  $G[S]$  is the subgraph of  $G$  *induced* by  $S$ ; that is,  $G[S]$  is a graph with vertex set  $S$  and edge set  $\{uv \in E(G) \mid \{u, v\} \subseteq S\}$ . For an integer  $n \geq 1$ , let  $K_n$  denote the complete graph of order  $n$ ; i.e.,  $K_n$  is an  $(n - 1)$ -regular graph of order  $n$ . For any function  $f : V(G) \rightarrow \mathbb{R}$ , we write  $f(S) = \sum_{v \in S} f(v)$  for all  $S \subseteq V(G)$ , and the *weight* of  $f$  is  $w(f) = f(V(G))$ .

Domination is an important subject in graph theory, and has numerous applications in other fields; see [11, 12] for comprehensive treatment and detailed surveys on (earlier) results in domination theory from both theoretical and applied perspectives. A set  $S \subseteq V(G)$  is called a *dominating set* (resp. *total dominating set*) of  $G$  if  $\bigcup_{v \in S} N_G[v] = V(G)$  (resp.  $\bigcup_{v \in S} N_G(v) = V(G)$ ). The *domination number* (resp. *total domination number*) of  $G$ , denoted by  $\gamma(G)$  (resp.  $\gamma_t(G)$ ), is the minimum size of a dominating set (resp. total dominating set) of  $G$ .

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Let  $k \geq 1$  be a fixed integer and  $G$  be a graph of minimum degree at least  $k - 1$ . A function  $f : V(G) \rightarrow \{-1, 1\}$  is called a *signed  $k$ -dominating function* of  $G$  if  $f(N_G[v]) \geq k$  for all  $v \in V(G)$ . The *signed  $k$ -domination number* of  $G$ , denoted by  $\gamma_{kS}(G)$ , is the minimum weight of a signed  $k$ -dominating function of  $G$ . When  $G$  is of minimum degree at least  $k$ , the *signed total  $k$ -dominating function* and *signed total  $k$ -domination number* of  $G$  (denoted by  $\gamma_{kS}^t(G)$ ) can be analogously defined by changing the closed neighborhood  $N_G[v]$  to the open neighborhood  $N_G(v)$  in the definition. The concepts of signed  $k$ -domination number and signed total  $k$ -domination number are introduced in [16], where sharp lower bounds of these numbers are established for general graphs, bipartite graphs and  $r$ -regular graphs in terms of the order of the graphs. A related graph parameter called the *upper signed  $k$ -domination number* of  $G$ , denoted by  $\Gamma_{kS}(G)$ , is defined in [17] as the maximum weight of a *minimal* signed  $k$ -dominating function of  $G$ . (A signed  $k$ -dominating function  $f$  of  $G$  is called *minimal* if there exists no signed  $k$ -dominating function  $f'$  of  $G$  such that  $f' \neq f$  and  $f'(v) \leq f(v)$  for every  $v \in V(G)$ .) This parameter has also been studied in [3].

In the special case where  $k = 1$ , the signed  $k$ -domination number and signed total  $k$ -domination number are exactly the *signed domination number* [5] and *signed total domination number* [18], respectively. These two parameters have been extensively studied in the literature; see e.g. [1, 2, 5, 6, 7, 9, 13, 14, 18, 19] and the references therein.

In this paper, we continue the investigation of the signed  $k$ -domination number and signed total  $k$ -domination number of graphs, from both algorithmic complexity and graph theoretic points of view. In Section 2 we show that, for every fixed  $k \geq 1$ , the problems of computing the signed  $k$ -domination number, the signed total  $k$ -domination number, and the upper signed  $k$ -domination number of a graph are all  $\mathcal{NP}$ -hard. We then present, in Section 3, sharp lower bounds on the signed  $k$ -domination number and signed total  $k$ -domination number for general graphs in terms of their minimum and maximum degrees, from which several interesting results follow immediately.

## 2 Complexity Issues of Signed (Total) $k$ -Domination

In this section we first show the  $\mathcal{NP}$ -hardness of computing the signed  $k$ -domination number and signed total  $k$ -domination number of a graph for all  $k \geq 1$ . Since the proofs for the two parameters are very similar, we only detail the proof for the signed total  $k$ -domination number, and merely point out the changes that need to be made for establishing hardness for the signed  $k$ -domination number. We now formally define the two decision problems corresponding to the computation of these two graph parameters.

SIGNED  $k$ -DOMINATION PROBLEM (SkDP)

*Instance:* A graph  $G = (V, E)$  and an integer  $r$ .

*Question:* Is  $\gamma_{kS}(G) \leq r$ ?

SIGNED TOTAL  $k$ -DOMINATION PROBLEM (STkDP)

*Instance:* A graph  $G = (V, E)$  and an integer  $r$ .

*Question:* Is  $\gamma_{kS}^t(G) \leq r$ ?

**Theorem 1.** *For every integer  $k \geq 1$ , the STkDP problem is  $\mathcal{NP}$ -complete.*

*Proof.* Let  $k \geq 1$  be a fixed integer. The STkDP problem is clearly in  $\mathcal{NP}$ . We now present a polynomial-time reduction from MINIMUM TOTAL DOMINATING SET (MTDS), which is a classical

$\mathcal{NP}$ -complete problem [8], to  $SkDP$ . The MTDS problem is defined as follows: Given a graph  $G$  and an integer  $r$ , decide whether  $G$  has a total dominating set of size at most  $r$ .

Let  $(G, r)$  be an instance of the MTDS problem. Construct another graph  $H$  as follows. First let  $H$  contain of a copy of  $G$ , which is denoted by  $G'$ . Also, for each vertex  $v \in V(G)$ , let  $v'$  denote its counterpart in  $G'$ . For each  $v \in V(G)$ , we add  $t(v)$  disjoint copies of  $K_{k+2}$  to  $H$ , where  $t(v) = d_G(v) + k - 2$ ; call these copies  $K_{k+2}^{v,1}, K_{k+2}^{v,2}, \dots, K_{k+2}^{v,t(v)}$ . Then, for each  $i \in \{1, 2, \dots, t(v)\}$ , add an edge between  $v'$  and an (arbitrary) vertex from  $K_{k+2}^{v,i}$ . This finishes the construction of  $H$ . It is easy to verify that  $d_H(v') = 2d_G(v) + k - 2$  for all  $v \in V(G)$ .

Let  $T = (k + 2) \sum_{v \in V(G)} t(v) = (k + 2) \sum_{v \in V(G)} (k + d_G(v) - 2)$  be the number of vertices in  $V(H \setminus G')$ . We will prove that  $\gamma_t(G) \leq r$  if and only if  $\gamma_{kS}^t(H) \leq 2r - |V(G)| + T$ .

First consider the “if” direction. Assume that  $\gamma_{kS}^t(H) \leq 2r - |V(G)| + T$ , and  $f : V(H) \rightarrow \{-1, 1\}$  is a signed total  $k$ -dominating function of  $H$  of weight  $\gamma_{kS}^t(H)$ . Let  $S' = \{v' \in V(G') \mid f(v') = 1\}$ . It is easy to see that, for each  $v \in V(G)$  and  $1 \leq i \leq t(v)$ , all vertices in  $K_{k+2}^{v,i}$  must have function value “1” under  $f$ . It follows that  $\gamma_{kS}^t(H) = w(f) = T + |S'| - (|V(G')| - |S'|) = 2|S'| - |V(G)| + T$ . Since  $\gamma_{kS}^t(H) \leq 2r - |V(G)| + T$ , we have  $|S'| \leq r$ . Now define  $S = \{v \in V(G) \mid v' \in S'\}$ ; i.e.,  $S$  is the counterpart of  $S'$  in  $G$ . We show that  $S$  is a total dominating set of  $G$ . Assume to the contrary that  $S$  is not a total dominating set of  $G$ , and let  $v \in V(G)$  be such that  $N_G(v) \cap S = \emptyset$ . By our definitions of  $S$  and  $S'$ ,  $f(u') = -1$  for all  $u \in N_G(v)$ . Thus,  $\sum_{x \in N_H(v')} f(x) \leq t(v) - d_G(v) = k - 2$ , contradicting with the fact that  $f$  is a signed total  $k$ -dominating function of  $H$ . Therefore,  $S'$  is indeed a total dominating set of  $G$ , from which  $\gamma_t(G) \leq |S'| \leq r$  follows. This completes the proof for the “if” direction.

Now comes the “only if” part of the reduction. Suppose  $\gamma_t(G) \leq r$  and  $S \subseteq V(G)$  is a total dominating set of  $G$  of size at most  $r$ . Define a function  $f : V(H) \rightarrow \{-1, 1\}$  as follows:  $f(x) = -1$  if  $x = v'$  for some  $v \in V(G) \setminus S$ , and  $f(x) = 1$  otherwise. The weight of  $f$  is  $T + |S| - (|V(G')| - |S|) = 2|S| - |V(G)| + T \leq 2r - |V(G)| + T$ . We now verify that  $f$  is a signed total  $k$ -dominating function of  $H$ . For each  $x \in V(H \setminus G')$ ,  $f(N_H(x)) \geq (k + 1) - 1 = k$ . For each  $v' \in V(G')$  (with  $v \in V(G)$ ), since  $S$  is a total dominating set of  $G$ ,  $f(N_H(v')) \geq t(v) + 1 - (d_G(v) - 1) = t(v) + 2 - d_G(v) = k$ . Hence,  $f$  is a signed total  $k$ -dominating function of  $H$  of weight at most  $2r - |V(G)| + T$ . This completes the “only if” part of the reduction.

Therefore,  $\gamma_t(G) \leq r$  if and only if  $\gamma_{kS}^t(H) \leq 2r - |V(G)| + T$ . This finishes the whole reduction, and hence concludes the proof of Theorem 1.  $\square$

**Theorem 2.** *For every integer  $k \geq 1$ , the  $SkDP$  problem is  $\mathcal{NP}$ -complete.*

*Proof.* The proof is very similar to that of Theorem 1, with two differences in the reduction. Therefore, we only describe the reduction. We reduce from the  $\mathcal{NP}$ -complete problem MINIMUM DOMINATING SET (which, given a graph  $G$  and an integer  $r$ , needs to decide whether  $G$  has a dominating set of size at most  $r$ ) to  $SkDP$ . Let  $(G, r)$  be an instance of MINIMUM DOMINATING SET. Construct another graph  $H$  as follows. First let  $H$  contain of a copy of  $G$ , which is denoted by  $G'$ . For each vertex  $v \in V(G)$ , add  $s(v)$  disjoint copies of  $K_{k+1}$  to  $H$ , where  $s(v) = d_G(v) + k - 1$ ; call these copies  $K_{k+1}^{v,1}, K_{k+1}^{v,2}, \dots, K_{k+1}^{v,s(v)}$ . Then, for each  $i \in \{1, 2, \dots, s(v)\}$ , add an edge between  $v'$  (the counterpart of  $v$  in  $G'$ ) and an arbitrary vertex from  $K_{k+1}^{v,i}$ . This finishes the construction of  $H$ . Using similar argument to that in Theorem 1, we can prove that  $\gamma(G) \leq r$  if and only if  $\gamma_{kS}(H) \leq 2r - |V(G)| + T$ , where  $T = (k + 1) \sum_{v \in V(G)} s(v)$ . The  $\mathcal{NP}$ -completeness of  $SkDP$  is thus established.  $\square$

We now define the problem corresponding to the computation of the upper signed  $k$ -domination number of graphs as follows.

UPPER SIGNED  $k$ -DOMINATION PROBLEM (US $k$ DP)

*Instance:* A graph  $G = (V, E)$  and an integer  $r$ .

*Question:* Is  $\Gamma_{kS}(G) \geq r$ ?

**Theorem 3.** *For every integer  $k \geq 1$ , the US $k$ DP problem is  $\mathcal{NP}$ -complete.*

*Proof.* The US $k$ DP problem is in  $\mathcal{NP}$  because given a function  $f : V(G) \rightarrow \{-1, 1\}$ , we can verify in polynomial time whether  $f$  is a minimal signed  $k$ -dominating function of  $G$  using Lemma 4 in [3]. We will describe a polynomial time reduction from the 1-in-3 SAT problem to it. The 1-in-3 SAT problem is defined as follows: Given a Boolean formula in conjunctive normal form, each clause of which contains exactly three *positive* literals (i.e., variables with no negations), decide whether the formula is *1-in-3 satisfiable*, i.e., if there exists an assignment of the variables such that exactly one variable of each clause is assigned TRUE. This problem is known to be  $\mathcal{NP}$ -complete [15].

Let  $F$  be a Boolean formula with variables  $\{x_1, x_2, \dots, x_n\}$ , which is an input of the 1-in-3 SAT problem. Assume  $F = \bigwedge_{i=1}^m c_i$  where  $c_i = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$  for each  $i \in \{1, 2, \dots, m\}$ . We construct a graph  $G$  as follows. Take  $m$  disjoint copies of  $K_{k+2}$ , each of which corresponds to a clause  $c_i$  with  $i \in \{1, 2, \dots, m\}$ , and  $n$  disjoint copies of  $K_{k+3}$  (also disjoint from the copies of  $K_{k+2}$ 's) each of which corresponds to a variable  $x_j$  with  $j \in \{1, 2, \dots, n\}$ . Delete one edge from each copy of  $K_{k+3}$ . We will call the copy of  $K_{k+2}$  corresponding to  $c_i$  the  *$i$ -th clause block*, and call the copy of  $K_{k+3}$  (with one edge missing) corresponding to  $x_j$  the  *$j$ -th variable block*. For each  $i \in \{1, 2, \dots, m\}$ , let  $c'_i$  be an (arbitrary) vertex in the  $i$ -th clause block. For every  $j \in \{1, 2, \dots, n\}$ , let  $x'_j$  and  $x''_j$  be the two vertices in the  $j$ -th variable block for which the edge  $x'_j x''_j$  is removed. For each clause  $c_i = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$ , add three cross-block edges  $c'_i x'_{i_1}$ ,  $c'_i x'_{i_2}$ , and  $c'_i x'_{i_3}$ . This finishes the construction of  $G$ . Note that  $|V(G)| = (k+3)n + (k+2)m$ .

We claim that  $\Gamma_{kS}(G) \geq (k+1)n + (k+2)m$  if and only if  $F$  is 1-in-3 satisfiable. First consider the “if” direction, and let  $\mathcal{A} : \{x_1, x_2, \dots, x_n\} \rightarrow \{\text{TRUE}, \text{FALSE}\}$  be an assignment that witnesses the 1-in-3 satisfiability of  $F$ . Define  $f : V(G) \rightarrow \{-1, 1\}$  as follows: For each  $j \in \{1, 2, \dots, n\}$ , let

$$f(x'_j) = \begin{cases} 1 & \text{if } \mathcal{A}(x_j) = \text{TRUE}; \\ -1 & \text{if } \mathcal{A}(x_j) = \text{FALSE} \end{cases} \quad \text{and} \quad f(x''_j) = \begin{cases} -1 & \text{if } \mathcal{A}(x_j) = \text{TRUE}; \\ 1 & \text{if } \mathcal{A}(x_j) = \text{FALSE}. \end{cases}$$

Let  $f(v) = 1$  for all  $v \in V(G) \setminus \bigcup_{j=1}^n \{x'_j, x''_j\}$ .

Clearly,  $w(f) = (k+1)n + (k+2)m$ . Since exactly one of  $\mathcal{A}(x_{i_1})$ ,  $\mathcal{A}(x_{i_2})$  and  $\mathcal{A}(x_{i_3})$  is TRUE for each  $1 \leq i \leq m$ , it is easy to verify that  $f$  is a signed  $k$ -dominating function of  $G$ . We next prove that  $f$  is minimal, that is, for every vertex  $v \in V(G)$  with  $f(v) = 1$  there exists  $u \in N_G[v]$  for which  $f(N_G[u]) \in \{k, k+1\}$  (see [3]). For every  $j \in \{1, 2, \dots, n\}$ , there is (at least) one vertex  $u$  in the  $j$ -th variable block such that  $u \notin \{x'_j, x''_j\}$ . This vertex  $u$  is adjacent to all other vertices in the  $j$ -th variable block, and clearly  $f(N_G[u]) = k+1$ . For every  $i \in \{1, 2, \dots, m\}$ ,  $c'_i$  is adjacent to all other vertices in the  $i$ -th clause block, and  $f(N_G[c'_i]) = (k+2) + (1-2) = k+1$  since exactly one of  $f(x'_{i_1})$ ,  $f(x'_{i_2})$  and  $f(x'_{i_3})$  is 1. Therefore,  $f$  is indeed a minimal signed  $k$ -dominating function of  $G$  with weight  $(k+1)n + (k+2)m$ , and the correctness of the “if” direction follows.

We now turn to the “only if” part of the claim. Assume that  $f$  is a minimal signed  $k$ -dominating function of  $G$  of weight at least  $(k+1)n + (k+2)m$ . If for some  $j \in \{1, 2, \dots, n\}$ , the vertices in the  $j$ -th variable block all have value 1 under  $f$ , then  $f(N_G[v]) \geq k+2$  for every  $v \neq x'_j$  in the  $j$ -th

variable block. Thus, there is no  $u \in N_G[x_j'']$  such that  $f(N_G[u]) \in \{k, k+1\}$ , which violates the minimality of  $f$ . Hence, at least one vertex from each variable block must have value  $-1$  under  $f$ , implying that  $w(f) \leq (k+1)n + (k+2)m$ . We thus have  $w(f) = (k+1)n + (k+2)m$ , and therefore (1)  $f(v) = 1$  for every vertex  $v$  in the clause blocks, and (2) for each  $j \in \{1, 2, \dots, n\}$ ,  $f(v) = -1$  for exactly one vertex  $v$  in the  $j$ -th variable block. Now produce an assignment  $\mathcal{A}$  as follows: For each  $j \in \{1, 2, \dots, n\}$ , let  $\mathcal{A}(x_j) = \text{TRUE}$  if  $f(x_j') = 1$ , and  $\mathcal{A}(x_j) = \text{FALSE}$  otherwise. For every  $i \in \{1, 2, \dots, m\}$ , we have  $k \leq f(N_G[c_i']) = (k+2) + f(x_{i_1}) + f(x_{i_2}) + f(x_{i_3})$ , and thus at least one of  $f(x_{i_1}), f(x_{i_2})$  and  $f(x_{i_3})$  must be 1. Assume that at least two of the three values are 1. Then  $f(N_G[c_i']) \geq k+3$ , and obviously  $f(N_G[v]) = k+2$  for every other vertex  $v$  in the  $i$ -th clause block. This indicates, however, that a vertex  $v \neq c_i'$  in the  $i$ -th clause block does not have any neighbor (including itself) whose closed-neighborhood-sum is  $k$  or  $k+1$ , contradicting with the minimality of  $f$ . Accordingly, exactly one of  $f(x_{i_1}), f(x_{i_2})$  and  $f(x_{i_3})$  is 1, and thus exactly one of  $\mathcal{A}(x_{i_1}), \mathcal{A}(x_{i_2})$  and  $\mathcal{A}(x_{i_3})$  is TRUE, for every  $i \in \{1, 2, \dots, m\}$ . Therefore,  $F$  is 1-in-3 satisfiable, finishing the proof of the “only if” part of the reduction.

The reduction is completed and the  $\mathcal{NP}$ -completeness of  $\text{USkDP}$  is thus established.  $\square$

### 3 Sharp Lower Bounds on $\gamma_{kS}(G)$ and $\gamma_{kS}^t(G)$

In this section we present sharp lower bounds on  $\gamma_{kS}(G)$  and  $\gamma_{kS}^t(G)$  in terms of the minimum and maximum degrees of  $G$ . Let  $k \geq 1$  be a fixed integer throughout this section. For each integer  $n$ , define  $I_n = 1$  if  $n \equiv k \pmod{2}$ , and  $I_n = 0$  otherwise; that is,  $I_n$  is the indicator variable of whether  $n$  and  $k$  have the same parity.

**Theorem 4.** *For every graph  $G$  with  $\delta(G) \geq k-1$ ,*

$$\gamma_{kS}(G) \geq |V(G)| \cdot \frac{\delta(G) - \Delta(G) + 2k + I_{\delta(G)} + I_{\Delta(G)}}{\delta(G) + \Delta(G) + 2 + I_{\delta(G)} - I_{\Delta(G)}}.$$

*Proof.* Let  $G$  be a graph of order  $n$  with  $\delta(G) \geq k-1$ . For notational simplicity, we write  $\delta$  and  $\Delta$  to denote  $\delta(G)$  and  $\Delta(G)$  respectively. When  $\delta = \Delta$ , it is easy to verify that the theorem degenerates to Theorem 5 in [16]. Thus, we assume in what follows that  $\Delta \geq \delta + 1$ . Let  $f$  be a signed  $k$ -dominating function of  $G$  of weight  $\gamma_{kS}(G)$ . We need to introduce some notations. Let  $P = \{v \in V(G) \mid f(v) = 1\}$  and  $Q = V(G) \setminus P = \{v \in V(G) \mid f(v) = -1\}$ . Furthermore, denote  $P_\delta = \{v \in P \mid d_G(v) = \delta\}$ ,  $P_\Delta = \{v \in P \mid d_G(v) = \Delta\}$ , and  $P_m = P \setminus (P_\delta \cup P_\Delta)$ . Define  $Q_\delta$ ,  $Q_\Delta$ , and  $Q_m$  analogously. For each  $c \in \{\delta, \Delta, m\}$ , let  $V_c = P_c \cup Q_c$ . Notice that  $V_\delta \cap V_\Delta = \emptyset$  since  $\Delta > \delta$ . Let  $R = \{v \in V(G) \mid d_G(v) \equiv k \pmod{2}\}$ . Clearly  $\sum_{y \in N_G[x]} f(y) \geq k+1$  for each  $x \in R$ .

Thus, we have

$$\begin{aligned}
kn + |R| &\leq \sum_{x \in V(G)} \sum_{y \in N_G[x]} f(y) = \sum_{x \in V(G)} (d_G(x) + 1)f(x) \\
&= (\delta + 1)|P_\delta| + (\Delta + 1)|P_\Delta| + \sum_{x \in P_m} (d_G(x) + 1) - (\delta + 1)|Q_\delta| - (\Delta + 1)|Q_\Delta| - \sum_{x \in Q_m} (d_G(x) + 1) \\
&\leq (\delta + 1)|P_\delta| + (\Delta + 1)|P_\Delta| + \Delta|P_m| - (\delta + 1)|Q_\delta| - (\Delta + 1)|Q_\Delta| - (\delta + 2)|Q_m| \\
&\quad (\text{since } \delta + 1 \leq d_G(x) \leq \Delta - 1 \text{ for each } x \in P_m \cup Q_m) \\
&= (\delta + 1)|V_\delta| + (\Delta + 1)|V_\Delta| + \Delta|V_m| - 2(\delta + 1)|Q_\delta| - 2(\Delta + 1)|Q_\Delta| - (\Delta + \delta + 2)|Q_m| \\
&= (\Delta + 1)n - (\Delta - \delta)|V_\delta| - |V_m| - (\Delta + \delta + 2)|Q| + (\Delta - \delta)|Q_\delta| - (\Delta - \delta)|Q_\Delta| \\
&\quad (\text{note that } n = |V(G)| = |V_\delta| + |V_\Delta| + |V_m| \text{ and } |Q| = |Q_\delta| + |Q_\Delta| + |Q_m|).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\Delta + 1 - k)n &\geq |R| + |V_m| + (\Delta - \delta)(|V_\delta| - |Q_\delta| + |Q_\Delta|) + (\Delta + \delta + 2)|Q| \\
&= |R| + |V_m| + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta + 2)|Q|.
\end{aligned}$$

Since  $R = \{v \in V(G) \mid d(v) \equiv k \pmod{2}\}$ , it holds that  $V_\delta \subseteq R$  if  $\delta \equiv k \pmod{2}$ , and that  $V_\Delta \subseteq R$  if  $\Delta \equiv k \pmod{2}$ . Recalling that  $V_\Delta \cap V_\delta = \emptyset$ , we have  $|R| \geq I_\delta \cdot |V_\delta| + I_\Delta \cdot |V_\Delta|$ . Thus,

$$\begin{aligned}
(\Delta + 1 - k)n &\geq I_\delta \cdot |V_\delta| + I_\Delta \cdot |V_\Delta| + |V_m| + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta + 2)|Q| \\
&= I_\Delta(|V_m| + |V_\delta| + |V_\Delta|) + (1 - I_\Delta)|V_m| + (I_\delta - I_\Delta)|V_\delta| \\
&\quad + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta + 2)|Q| \\
&= I_\Delta \cdot n + (1 - I_\Delta)|V_m| + (I_\delta - I_\Delta)|V_\delta| + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta + 2)|Q|.
\end{aligned}$$

Observing that  $\Delta - \delta \geq 1 \geq \max\{I_\delta - I_\Delta, I_\Delta - I_\delta\}$  and  $(1 - I_\Delta)|V_m| \geq (1 - I_\Delta)|Q_m| \geq (I_\delta - I_\Delta)|Q_m|$ , we get

$$\begin{aligned}
&(\Delta + 1 - k - I_\Delta)n \\
&\geq (I_\delta - I_\Delta)|Q_m| + (I_\delta - I_\Delta)|V_\delta| + (I_\Delta - I_\delta)|P_\delta| + (I_\delta - I_\Delta)|Q_\Delta| + (\Delta + \delta + 2)|Q| \\
&= (I_\delta - I_\Delta)(|Q_m| + |V_\delta| - |P_\delta| + |Q_\Delta|) + (\Delta + \delta + 2)|Q| \\
&= (I_\delta - I_\Delta)(|Q_m| + |Q_\delta| + |Q_\Delta|) + (\Delta + \delta + 2)|Q| \\
&= (\Delta + \delta + 2 + I_\delta - I_\Delta)|Q|.
\end{aligned}$$

Hence, we deduce that

$$|Q| \leq n \cdot \frac{\Delta - k + 1 - I_\Delta}{\delta + \Delta + 2 + I_\delta - I_\Delta},$$

from which it follows that

$$\gamma_{kS}(G) = n - 2|Q| \geq n \cdot \frac{\delta - \Delta + 2k + I_\delta + I_\Delta}{\delta + \Delta + 2 + I_\delta - I_\Delta},$$

which is exactly the desired inequality in Theorem 4.  $\square$

A vertex of degree  $k - 1$  or  $k$  in a graph  $G$  clearly has function value 1 under all signed  $k$ -dominating functions of  $G$ . Thus, it is natural to consider graphs with minimum degree at least  $k + 1$  (as is done in [3] for establishing sharp upper bounds for the upper signed  $k$ -domination number). We next show that Theorem 4 is sharp for all  $\Delta \geq \delta \geq k + 1$ . This level of sharpness is high as it applies not only to special values of minimum and maximum degrees.

**Theorem 5.** *For any integers  $\delta$  and  $\Delta$  such that  $\Delta \geq \delta \geq k + 1$ , there exists an infinite family  $\mathcal{F}$  of graphs with minimum degree  $\delta$  and maximum degree  $\Delta$ , such that for every graph  $G \in \mathcal{F}$ ,*

$$\gamma_{kS}(G) = |V(G)| \cdot \frac{\delta - \Delta + 2k + I_\delta + I_\Delta}{\delta + \Delta + 2 + I_\delta - I_\Delta}.$$

*Proof.* Fix integers  $\Delta$  and  $\delta$  such that  $\Delta \geq \delta \geq k + 1$ . Let  $H_1, H_2, \dots, H_t$  be  $t$  disjoint copies of the complete bipartite graph  $K_{a,b}$  with vertex partition  $(A, B)$ , where  $|A| = a = (\delta + k + 1 + I_\delta)/2$ ,  $|B| = b = (\Delta - k + 1 - I_\Delta)/2$  (it is easy to verify that  $a$  and  $b$  are both integers), and  $t$  is an arbitrary even integer larger than  $\Delta$ . It is also easy to check that  $1 \leq a \leq \delta$  and  $1 \leq b \leq \Delta$  (just note that  $I_\delta = 0$  when  $\delta = k + 1$ ). For each  $1 \leq i \leq t$ , let  $A_i$  and  $B_i$  denote the vertex partition of  $H_i$  with size  $a$  and  $b$ , respectively. Let  $P = \bigcup_{i=1}^t A_i$  and  $Q = \bigcup_{i=1}^t B_i$ . Note that each vertex in  $P$  is connected to exactly  $b$  vertices in  $Q$ , and each vertex in  $Q$  is adjacent to exactly  $a$  vertices in  $P$ .

Our desired graph  $G$  has vertex set  $P \cup Q$ , and contains  $\bigcup_{i=1}^t H_i$  as a subgraph. Furthermore, we add some edges between vertices in  $P$  to make  $G[P]$  become  $(\Delta - b)$ -regular (no edges need to be added if  $\Delta = b$ ). This can be done in the following way: Imagine that there is a complete graph  $K$  whose vertex set is  $P$ . Since  $|P| = ta$  is even and every complete graph of even order is 1-factorable (see e.g. Theorem 9.1 in [10]), the edges of  $K$  can be partitioned into  $|P| - 1 \geq \Delta$  perfect matchings of  $K$ . Taking  $\Delta - b$  of these matchings and adding them to  $G$  certainly makes  $G[P]$  become  $(\Delta - b)$ -regular. Similarly, we add some edges between vertices in  $Q$  to make  $G[Q]$   $(\delta - a)$ -regular. This finishes the construction of  $G$ . Note that all vertices in  $P$  have degree  $\Delta$  and those in  $Q$  have degree  $\delta$ , and thus  $G$  is of minimum degree  $\delta$  and maximum degree  $\Delta$ . (Note also that by varying  $t$ , we get an infinite family of graphs with the desired properties.)

Define a function  $f : P \cup Q \rightarrow \{-1, 1\}$  by letting  $f(v) = 1$  for all  $v \in P$  and  $f(u) = -1$  for all  $u \in Q$ . Then, for each  $v \in P$ ,  $f(N_G[v]) = \Delta + 1 - 2b = k + I_\Delta \geq k$ , and for each  $u \in Q$ ,  $f(N_G[u]) = 2a - (\delta + 1) = k + I_\delta \geq k$ . Therefore,  $f$  is a signed  $k$ -dominating function of  $G$ . Since  $|V(G)| = |P| + |Q|$  and  $|P|/|Q| = a/b = \frac{\delta + k + 1 + I_\delta}{\Delta - k + 1 - I_\Delta}$ , we have

$$\gamma_{kS}(G) \leq w(f) = |P| - |Q| = \left(1 - \frac{2}{|P|/|Q| + 1}\right)|V(G)| = |V(G)| \cdot \frac{\delta - \Delta + 2k + I_\delta + I_\Delta}{\delta + \Delta + 2 + I_\delta - I_\Delta}.$$

By Theorem 4, we know that the equality holds in the above formula, which completes the proof of Theorem 5.  $\square$

We can also derive a sharp lower bound on the signed total  $k$ -domination number of a graph as follows.

**Theorem 6.** *For every graph  $G$  with  $\delta(G) \geq k$ ,*

$$\gamma_{kS}^t(G) \geq |V(G)| \cdot \frac{\delta(G) - \Delta(G) + 2k + 2 - I_{\delta(G)} - I_{\Delta(G)}}{\delta(G) + \Delta(G) + I_{\Delta(G)} - I_{\delta(G)}}.$$

**Theorem 7.** For any integers  $\delta$  and  $\Delta$  such that  $\Delta \geq \delta \geq k + 2$ , there exists an infinite family  $\mathcal{F}$  of graphs with minimum degree  $\delta$  and maximum degree  $\Delta$ , such that for every graph  $G \in \mathcal{F}$ ,

$$\gamma_{kS}^t(G) = |V(G)| \cdot \frac{\delta - \Delta + 2k + 2 - I_\delta - I_\Delta}{\delta + \Delta + I_\Delta - I_\delta}.$$

The proofs of Theorems 6 and 7 are very similar to those of Theorems 4 and 5, and thus are put in the appendix.

Theorems 4 and 6 are generalizations of Theorem 5 in [16]. The following corollaries, which generalize some other known results regarding signed domination number and signed total domination number, are also immediate from the preceding theorems.

**Corollary 1.** For any nearly  $r$ -regular graph  $G$  of order  $n$  with  $r \geq k$ ,  $\gamma_{kS}(G) \geq kn/(r + I_{r-1})$  and  $\gamma_{kS}^t(G) \geq kn/(r - I_{r-1})$ .

**Corollary 2.** Let  $c$  be a real number for which  $-1 < c \leq 1$ . Then  $\gamma_{kS}(G) \geq cn$  for every graph  $G$  of order  $n$  with  $\delta(G) \geq k - 1$  and  $\Delta(G) \leq ((1 - c)\delta(G) + 2k - 2c)/(1 + c)$ , and  $\gamma_{kS}^t(G) \geq cn$  for every graph  $G$  of order  $n$  with  $\delta(G) \geq k$  and  $\Delta(G) \leq ((1 - c)\delta(G) + 2k)/(1 + c)$ .

**Corollary 3.** Let  $G$  be a graph with  $\delta(G) \geq k$  and  $\Delta(G) \leq \delta(G) + 2k$ . Then  $\gamma_{kS}(G) \geq 0$  and  $\gamma_{kS}^t(G) \geq 0$ .

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## A Proof of Theorem 6

*Proof of Theorem 6.* Let  $G$  be a graph of order  $n$  and  $f$  be a signed total  $k$ -dominating function of  $G$ . Let  $\delta, \Delta, P, Q, P_\delta, P_\Delta, P_m, Q_\delta, Q_\Delta, Q_m, V_\delta, V_\Delta, V_m$  be defined in the same way as in the proof of Theorem 4. Let  $R = \{v \in V(G) \mid d(v) \not\equiv k \pmod{2}\}$  (which is different from the definition of  $R$  in the proof of Theorem 4). Assume  $\Delta > \delta$ , otherwise the theorem just becomes Theorem 5 in [16]. Since  $\sum_{y \in N_G(x)} f(y) \geq k + 1$  for all  $x \in R$ , we have:

$$\begin{aligned}
kn + |R| &\leq \sum_{x \in V(G)} \sum_{y \in N_G(x)} f(y) \\
&= \sum_{x \in V(G)} d_G(x) f(x) \\
&= \delta |P_\delta| + \Delta |P_\Delta| + \sum_{x \in P_m} d_G(x) - \delta |Q_\delta| - \Delta |Q_\Delta| - \sum_{x \in Q_m} d_G(x) \\
&\leq \delta |P_\delta| + \Delta |P_\Delta| + (\Delta - 1) |P_m| - \delta |Q_\delta| - \Delta |Q_\Delta| - (\delta + 1) |Q_m| \\
&= \delta |V_\delta| + \Delta |V_\Delta| + (\Delta - 1) |V_m| - 2\delta |Q_\delta| - 2\Delta |Q_\Delta| - (\Delta + \delta) |Q_m| \\
&= \Delta n - (\Delta - \delta) |V_\delta| - |V_m| - (\Delta + \delta) |Q| + (\Delta - \delta) |Q_\delta| - (\Delta - \delta) |Q_\Delta| \\
&\quad (\text{recall that } n = |V(G)| = |V_\delta| + |V_\Delta| + |V_m| \text{ and } |Q| = |Q_\delta| + |Q_\Delta| + |Q_m|).
\end{aligned}$$

By our definition, it holds that  $|R| \geq (1 - I_\delta)|V_\delta| + (1 - I_\Delta)|V_\Delta|$ . Therefore,

$$\begin{aligned}
(\Delta - k)n &\geq |R| + |V_m| + (\Delta - \delta)(|V_\delta| - |Q_\delta| + |Q_\Delta|) + (\Delta + \delta)|Q| \\
&= |R| + |V_m| + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta)|Q| \\
&\geq (1 - I_\delta)|V_\delta| + (1 - I_\Delta)|V_\Delta| + |V_m| + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta)|Q| \\
&= (1 - I_\Delta)(|V_m| + |V_\delta| + |V_\Delta|) + I_\Delta|V_m| + (I_\Delta - I_\delta)|V_\delta| \\
&\quad + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta)|Q| \\
&= (1 - I_\Delta)n + I_\Delta|V_m| + (I_\Delta - I_\delta)|V_\delta| + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta)|Q|.
\end{aligned}$$

Noting that  $I_\Delta|V_m| \geq (I_\Delta - I_\delta)|Q_m|$  and  $\Delta - \delta \geq \max\{I_\Delta - I_\delta, I_\delta - I_\Delta\}$ , we obtain

$$\begin{aligned}
&(\Delta - k + I_\Delta - 1)n \\
&\geq I_\Delta|V_m| + (I_\Delta - I_\delta)|V_\delta| + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta)|Q| \\
&\geq (I_\Delta - I_\delta)|Q_m| + (I_\Delta - I_\delta)|V_\delta| + (I_\delta - I_\Delta)|P_\delta| + (I_\Delta - I_\delta)|Q_\Delta| + (\Delta + \delta)|Q| \\
&= (I_\Delta - I_\delta)(|Q_m| + |V_\delta| - |P_\delta| + |Q_\Delta|) + (\Delta + \delta)|Q| \\
&= (I_\Delta - I_\delta)|Q| + (\Delta + \delta)|Q| \\
&= (\Delta + \delta + I_\Delta - I_\delta)|Q|.
\end{aligned}$$

Hence, we have

$$|Q| \leq n \cdot \frac{\Delta - k + I_\Delta - 1}{\delta + \Delta + I_\Delta - I_\delta},$$

from which it follows that

$$\gamma_{kS}(G) = n - 2|Q| \geq n \cdot \frac{\delta - \Delta + 2k + 2 - I_\delta - I_\Delta}{\delta + \Delta + I_\Delta - I_\delta},$$

completing the proof of Theorem 6.  $\square$

## B Proof of Theorem 7

*Proof of Theorem 7.* Fix integers  $\Delta$  and  $\delta$  such that  $\Delta \geq \delta \geq k + 2$ . We proceed with the same construction used in the proof of Theorem 5, except for setting  $a = (\delta + k - I_\delta + 1)/2$  and  $b = (\Delta - k + I_\Delta - 1)/2$  instead. (It is easy to check that  $a$  and  $b$  are integers satisfying that  $1 \leq a \leq \delta$  and  $1 \leq b \leq \Delta$ .) The obtained graph  $G$  has vertex set  $P \cup Q$ , where  $d_G(v) = \Delta$  for all  $v \in P$  and  $d_G(u) = \delta$  for all  $u \in Q$ . Furthermore, each vertex  $v \in P$  is adjacent to exactly  $b$  vertices in  $Q$  and  $\Delta - b$  vertices in  $P$ , while every vertex  $u \in Q$  is adjacent to precisely  $a$  vertices in  $P$  and  $\delta - a$  vertices in  $Q$ . Now define a function  $f$  which assigns 1 to all vertices in  $P$  and  $-1$  to those in  $Q$ . It is easy to verify that  $f$  is a signed total  $k$ -dominating function of  $G$  with weight  $|V(G)| \cdot \frac{\delta - \Delta + 2k + 2 - I_\delta - I_\Delta}{\delta + \Delta + I_\Delta - I_\delta}$ , completing the proof of Theorem 7.  $\square$